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CAUCHY PROBLEM FOR EQUATIONS OF INTERNAL WAVES*

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The Cauchy problem is considered for the equation of internal waves to which reduce many problems of the linear theory of waves in a continuously stratified fluid. The theorem of uniqueness is proved, and the formula for explicit representation of solution in terms of integrals whose kernels contain the obtained in /1/ fundamental solution of the internal wave operator and its time derivative are derived. Asymptotic analysis of solution in the "distant zone" is carried out for large values of dimensionless time.

The Cauchy problem for the equation defining the propagation of long gravitational waves in a rotating compressible barotropic fluid was first solved and its solution asymptotically analyzed by Obukhov /2/ in 1948. A singularity of the internal wave equation is that it is insoluble for the higher time deriative of the sought function. Sobolev, while investigating unsteady motions of a rotating fluid /3/, was the first to solve the Cauchy problem for an equation of this type which differed from the inner wave equation only be the substitution of the second-order derivative with respect to one three-dimensional variable for the two-dimensional Laplace operator. Sobolev's equation and certain of its extensions were considered in several papers /4-6/ et al). In /7,8/ the Cauchy problem was considered for a system of differential equations in partial derivatives that are insolvable for time derivatives of the unknown function.

Methods and results of indicated investigations are used below. Thus, the uniqueness theorem is formulated in conformity with that in /8/ with the refinement introduced in /4/, its proof is reduced to the test of fulfillment of conditions of uniqueness theorem for the equation of internal waves presented in /8/ but, also, with one refinement. A short account of some results of the paper are given in /9/.

1. Statement of the problem.

We define the operator $\mathbf{N}$ of internal waves as follows /1/:

$$\mathbf{N} = \frac{\partial^2}{\partial t^2}\Delta_3 + N^2\Delta_2 \tag{1.1}$$

where t is the time, Δ_3 is the three-dimensional Laplace operator of space coordinates x_1, x_2, x_3, Δ_2 is the two-dimensional Laplace operator of horizontal coordinates x_1, x_2, and N is the so-called Brunt-Väisälä frequency which defines density distribution of an inhomogeneous fluid in its unperturbed state. As in /1/, $N^2 = \text{const} > 0$ is assumed, which corresponds to the case when the density ρ_0 of the quiescent fluid depends only on the vertical coordinate x_3 directed against gravity acceleration in conformity with the law

$$\rho_0(x_3) = \rho_0(0)\exp(-N^2g^{-1}x_3)$$

Let us consider the classical Cauchy problem for the internal wave equation

$$\mathbf{N}u \equiv \frac{\partial^2}{\partial t^2}\Delta_3 u + N^2\Delta_2 u = f(x, t) \tag{1.2}$$

$$u|_{t=0} = u_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=1} = u_1(x) \tag{1.3}$$

where $f(x, t)$, $u_0(x)$ and $u_1(x)$ are specified functions, and $x = (x_1, x_2, x_3)$ is a point of the three-dimensional Euclidean space R^3.

2. The theorem of uniqueness of solution of the Cauchy problem.

Let the solution of problem (1.2), (1.3) with zero input data and zero right-hand side, whose derivative with respect to t and, also, the derivatives with respect to x_j of first and second order do not increase as $|x| \to \infty$ at a rate higher than $|x|^l$, where $l \geqslant 0$. Then that solution is a polynomial in x_1, x_2, x_3 of power not higher than l with coefficients that depend on t and vanish when $t = 0$.

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Proof. Setting in Eq. (1.2) $f = 0$ and introducing functions $w_1 = u$, $w_2 = -\partial u/\partial t$, we reduce it to the equivalent system of equations in $w_1(x, t)$ and $w_2(x, t)$

$$\frac{\partial w_1}{\partial t} = -w_2, \quad \frac{\partial}{\partial t}\Delta_3 w_2 = N^2 \Delta_2 w_1 \tag{2.1}$$

System (2.1) belongs to the class of systems considered in /8/. To make use of results of that work, we set, as in /8/.

$$w_k(x, t) = v_k(t; \xi)\exp[-i(x, \xi)],\ k = 1, 2$$
$$(x, \xi) = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$$

and obtain for the vector function $\mathbf{v} = (v_1, v_2)$ the system of ordinary differential equations

$$M(\xi)\frac{\partial \mathbf{v}}{\partial t} = L(\xi)\mathbf{v}$$
$$M = \begin{Vmatrix} 1 & 0 \\ 0 & |\xi|^2 \end{Vmatrix}, \quad L = \begin{Vmatrix} 0 & -1 \\ N^2(\xi_1^2 + \xi_2^2) & 0 \end{Vmatrix} \tag{2.2}$$

Applied to this system, the conditions imposed in the uniqueness theorem in /8/ must be such that for $t \in [0, T]$ the conditions must be satisfied:

1) Elements $v_{k,m}$ of the fundamental system of system (2.2) must satisfy the inequalities

$$|v_{k,m}| \leqslant A|\xi|^{-q},\ q \geqslant 0,\ |\xi| \leqslant 1 \tag{2.3}$$
$$|v_{k,m}| \leqslant A|\xi|^{p},\ p \geqslant 0,\ |\xi| > 1$$

where, and subsequently, A denotes arbitrary constants;

2) $\det M(\xi)$ must be a polynomial in ξ_j vanishing only when $|\xi| = 0$, and after the substitution $\xi_j = \xi_j'|\xi|$ can be represented in the form

$$\det M = |\xi|^c(a_0 + a_1|\xi| + \ldots + a_G|\xi|^G),\ a_0 a_G \neq 0 \tag{2.4}$$

3) The elements of matrix $\|\mu_{ik}\| = M^{-1}L$ must satisfy the inequalities

$$|\mu_{i,k}| \leqslant A|\xi|^{-s},\ |\xi| \leqslant 1;\ |\mu_{i,k}| \leqslant A|\xi|^r,\ |\xi| > 1 \tag{2.5}$$

For system (2.2) the inequalities (2.3) and (2.5) are satisfied.

Indeed, the fundamental system of solutions of system (2.2) is of the form

$$v_{1,1} = \cos \nu t,\ v_{1,2} = -\nu^{-1}\sin \nu t$$
$$v_{2,1} = \nu \sin \nu t,\ v_{2,2} = \cos \nu t$$
$$\nu = N|\xi|^{-1}(\xi_1^2 + \xi_2^2)^{1/2}$$

which implies (2.3) and $q = p = 0$.

Matrix $\|\mu_{ik}\|$ for system (2.2) is

$$\|\mu_{ik}\| = \begin{Vmatrix} 0 & -1 \\ \nu^2 & 0 \end{Vmatrix}$$

which implies (2.5) and $s = r = 0$.

The determinant of matrix $M(\xi)$ for system (2.2) is equal $|\xi|^2$ and, consequently, is a polynomial in ξ_j, that vanishes only when $|\xi| = 0$. It can, however, be represented in the form (2.4) with $a_0 a_G \neq 0$ by refining the formulation of condition 2) by explicitly indicating the admissibility of the case $G = 0$, since the proof of the theorem remains valid in this case.

Thus, all conditions imposed in the uniqueness theorem in /8/ are satisfied for system (2.1), moreover functions w_1 and w_2 belong to the class for which that theorem was proved. Thus the statement of the uniqueness theorem given in /8/ and refined in /4/ is also valid for (2.1). In the case of (2.1) it reduces to the statement that system (2.1) satisfies the following two systems of equations

$$\begin{cases} w_1 = P_1 \\ \Delta_3 w_2 = P_2 \end{cases}, \quad \begin{cases} -w_2 = \dfrac{\partial P_1}{\partial t} \\ N^2\Delta_2 w_1 = \dfrac{\partial P_2}{\partial t} \end{cases}$$

where $P_1(t, x)$ and $P_2(t, x)$ are polynomials in x_j of power not higher than l, with coefficients dependent on t, which satisfy the initial condition $P_j(0, x) = 0$.

To complete the proof of the uniqueness theorem for the internal wave equation we point out the solution of the equation considered here agrees with function w_1 (in the sense in which the latter was introduced).

Corollary. If the conditions of the theorem specify that the solution must approach zero as $|x| \to \infty$, while retaining previous requirements as regards the solution derivatives, the Cauchy problem (1.2), (1.3) with zero initial data and zero right-hand side has only a zero solution.

3. Solution of the Cauchy problem for the equation of internal waves. Let us assume that the right-hand side of Eq.(1.2) with initial data (1.3) have the following properties: function $f(x, t)$ is continuous in R^4 when $t \geqslant 0$, the products of functions $\Delta_3 u_0(x)$, $\Delta_3 u_1(x)$ and their first order derivatives are integrable on $1/|x|$, and that function $f(x, t)$ and its derivatives with respect to x_j have the latter property for every $t \geqslant 0$, and as $|x| \to \infty$ the relations

$$u_0(x) = o(1), \quad u_1(x) = o(1), \quad \frac{\partial u_0}{\partial |x|} = o(1), \quad \frac{\partial u_1}{\partial |x|} = o(1)$$

apply.

Let us derive the problem solution using the apparatus of the theory of generalized functions.

First, in conformity with the general scheme /10/ we formulate and solve the generalized Cauchy problem for Eq.(1.2).

Let us assume the existence of a classic solution $u(x, t)$ of problem (1.2), (1.3). We introduce functions $u^*(x, t)$ and $f^*(x, t)$ that coincide for $t \geqslant 0$, respectively, with $u(x, t)$ and $f(x, t)$ and are zero for $t < 0$.

In the space of generalized functions $D'(R^4)$ function $u^*(x, t)$ satisfies the equation

$$\mathbf{N} u^* = f^*(x, t) + \Delta_3 u_0(x) \times \delta'(t) + \Delta_3 u_1(x) \times \delta(t) \tag{3.1}$$

where the right-hand side contains direct products of functions $\Delta_3 u_0(x)$ and $\Delta_3 u_1(x)$ by $\delta'(t)$ and $\delta(t)$.

Indeed, for all functions $\varphi(x, t)$ in the space of basic functions $D(R^4)$ we have the sequence of equalities

$$(\mathbf{N}u^*, \varphi) = (u^*, \mathbf{N}\varphi) = \lim_{\varepsilon \to +0} \Big[\int_\varepsilon^\infty \int_{R^3} \varphi \mathbf{N} u \, dx\, dt - \int_{R^3} \frac{\partial \varphi(x, \varepsilon)}{\partial t} \Delta_3 u(x, \varepsilon)\, dx + \int_{R^3} \varphi(x, \varepsilon) \frac{\partial}{\partial t} \Delta_3 u(x, \varepsilon) dx \Big]$$

By virtue of properties of functions $u(x, t)$ and $\varphi(x, t)$ we can set $\varepsilon = 0$ in expressions in brackets, and obtain

$$(Nu, \varphi) = (f^* + \Delta_3 u_0(x) \times \delta'(t) + \Delta_3 u_1(x) \times \delta(t), \varphi(x, t))$$

The generalized Cauchy problem for the operator of internal waves $\mathbf{N}$ with source $f^* \in D'(R^4)$ and initial perturbations $u_0(x) \in D'(R^3)$ and $u_1(x) \in D'(R^3)$ is understood here as the problem of finding the generalized function $u^*(x, t) \in D'(R^4)$ that vanishes for $t < 0$ and satisfies Eq.(3.1).

If functions $f(x, t)$, $u_0(x)$ and $u_1(x)$ are such that a convolution of the right-hand side of Eq.(3.1) with fundamental solution of operator $\mathbf{N}$ exists in $D'(R^4)$, a solution of the generalized Cauchy problem (3.1) exists in $D'(R^4)$ and is defined by the formula

$$u^*(x, t) = f^*(x, t) * E(x, t) + [\Delta_3 u_0(x) \times \delta(t)] * \frac{\partial E(x, t)}{\partial t} + [\Delta_3 u_1(x) \times \delta(t)] * E(x, t) \tag{3.2}$$

$$E(x, t) = -\frac{\theta(t)}{2\pi^2 N |x|} \int_{|x_3|/|x|}^1 \frac{\sin Ntu \, du}{(u^2 - x_3^2/|x|^2)^{1/2} (1 - u^2)^{1/2}}$$

where the symbol $*$ denotes convolution of functions, and E is the fundamental solution of the internal wave operator, derived in /1/.

If functions $u_0(x), u_1(x)$ and $f(x, t)$ possess the properties defined at the beginning of Sect.3, the unique solution of the classical Cauchy problem (1.2), (1.3) is defined by formula

$$u(x, t) = \int_0^t \int_{R^3} f(\xi, \tau) E(x - \xi, t - \tau)\, d\xi\, d\tau + \int_{R^3} \Delta_3 u_0(\xi) \frac{\partial E}{\partial t}(x - \xi, t)\, d\xi + \int_{R^3} \Delta_3 u_1(\xi) E(x - \xi, t)\, d\xi \tag{3.3}$$

To prove this we point out, first of all, that on the assumptions made above the right-hand side of formula (3.2) exists in $D'(R^4)$ and is expressed by formula (3.3), i.e. the latter is the solution of the generalized Cauchy problem (3.1), which holds without the stipulation of existence of third order derivatives in $u_0(x)$, $u_1(x)$ and of first order in $f(x, t)$.

Function $u(x, t)$ represented by formula (3.3) has for $t > 0$ continuous second order derivatives with respect to x_j which, in turn have second order derivatives with respect to t. Indeed, it is possible to differentiate once the integrands in (3.3) with respect to x_j, since the derivatives of $E(x - \xi, t)$ have integrable singularities of the type $|x - \xi|^{-2}$ and decrease as $|\xi| \to \infty$ but not slower than $A|\xi|^{-2}$. This property of the fundamental solution is deduced from the following representation(*):

$$E(x, t) = -\frac{\theta(t)}{2\pi^2 N |x|} \int_0^\infty \frac{\sin[Nt(v^2 + x_3^2/|x|^2)^{1/2}(v^2+1)^{-1/2}]\,dv}{(v^2 + x_3^2/|x|^2)^{1/2}(v^2+1)^{1/2}}$$

since differentiation of the integrand of the last integral with respect to x_j is admissible.

The integrands in (3.3) can be differentiated for the second time with respect to x_j, after the introduction of the new variable of integration $\eta = x - \xi$. Validity of this operation is based on the requirements imposed on $\Delta_3 u_0(x)$ and $\Delta_3 u_1(x)$. The differentiation of integrands in (3.3) with respect to t can be carried out any number of times. Hence the generalized solutions of Eq.(3.1) has for $t > 0$ the required number of classical derivatives for operator N; it is consequently, the classical solution of Eq.(3.1) when $t > 0$, hence, also, of Eq.(1.2). As $t \to +0$, function $u(x, t)$ satisfies the initial conditions (1.3), since the first integral in the right-hand side of (3.3) is at the limit zero, while it is possible to set $t = 0$ in the integrands of the other two integrals.

Hence the constructed function $u(x, t)$ is the solution of problem (1.2), (1.3).

Moreover, since function (3.3) approaches zero as $|x| \to \infty$ and its derivatives appear to be, at least, bounded, hence by virtue of the corollary of the proved uniqueness theorem, this function is the unique solution of the Cauchy problem (1.2), (1.3).

Remark. Since input data appear in formula (3.3) in terms of Laplace operators of initial functions, hence the indicated fundamental solution may be called the fundamental second order solution of the Cauchy problem considered here /4/. By analogy with /4/ we can show that such solution has a singularity only at the coordinate origin and is differentiable the required number of times outside that point and uniformly approaches zero with increasing distance from the coordinate origin, is unique.

4. Asymptotic representations of solution of the Cauchy problem and their hydrodynamic meaning. We shall derive the asymptotic representation of solution of the Cauchy problem (1.2), (1.3), assuming the right-hand side $f(x, t)$ of Eq.(1.2) equal zero. We use formula (3.3) and formula (7) from /9/ written in the form

$$u(x, t) = \int_{R^3} [Q_1(\xi)\exp(i\Phi_1) + Q_2(\xi)\exp(i\Phi_2)]\,d\xi \tag{4.1}$$

$$Q_j(\xi) = 16^{-1}\pi^{-3}\{F_x[u_0] + i(-1)^j F_x[u_1]/\nu(\xi)\}$$

$$\Phi_j = -(x, \xi) - (-1)^j \nu(\xi)\,t, \qquad F_x[u_j] = \int_{R^3} u_j(x)\exp\{i(x, \xi)\}\,dx$$

It is reasonable to assume that functions $u_0(x)$ and $u_1(x)$ satisfy the conditions of applicability of formulas used here.

Let us consider three cases assuming in the first and third of them that $u_0(x)$ and $u_1(x)$ are nonzero only in some region B of diameter d and containing inside the coordinate origin.

The distant zone. This zone contain points x for which $|x|/d \gg 1$, i.e., in so-called, the distant zone. The solution will be investigated for time t in the interval $[0, t_1]$ where $t_1 < \infty$.

We shall use formula (3.3) which can be represented in the form

$$u(x, t) = \frac{\partial E(x, t)}{\partial t} \int_B \Delta_3 u_0(\xi)\,d\xi + E(x, t)\int_B \Delta_3 u_1(\xi)\,d\xi + R\left(\frac{x}{d}, Nt\right) \tag{4.2}$$

$$|R| \leqslant [N|x|(|x|/d)^{1/4}]^{-1}\left[(2 + Nt_1)\frac{d^{1/4}N}{|x|^{1/4}}\int_B |\Delta_3 u_0(\xi)|\,d\xi + 4Nt_1\int_B |\Delta_3 u_1(\xi)|\,d\xi\right]$$

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*) V.A. Gorodtsov and E.V. Teodorovich, Linear internal waves and exponentially stratified perfect incompressible fluid. Preprint No.114, Inst. of Problems of Mechanics, Akad. Nauk SSSR, Moscow, 1978.

From the estimate (4.2) we obtain on the above assumptions the following approximate formula for the solution of the Cauchy problem in the distant zone

$$u(x,t) \sim \frac{\partial E(x,t)}{\partial t} \int_B \Delta_3 u_0(\xi)\, d\xi + E(x,t) \int_B \Delta_3 u_1(\xi)\, d\xi \tag{4.3}$$

Large values of Nt and fixed x. Using representation (4.1) and introducing new variables of integration τ, θ, φ by formulas

$$\begin{gathered}\xi_1 = \tau |x|^{-1} \sin\theta \cos(\varphi + \varphi_0), \quad \xi_2 = \tau |x|^{-1} \sin\theta \sin(\varphi + \varphi_0) \\ \xi_3 = \tau |x|^{-1} \cos\theta \\ \sin\varphi_0 = x_2/r, \quad \cos\varphi_0 = x_1/r, \quad r = (x_1^2 + x_2^2)^{1/2}\end{gathered} \tag{4.4}$$

we represent (4.1) thus:

$$u(x,t) = \frac{1}{8\pi^3} \int_0^\pi [f_0(\theta,x) \sin\theta \cos(Nt\sin\theta) + \frac{1}{N} f_1(\theta,x) \sin(Nt\sin\theta)]\, d\theta \tag{4.5}$$

$$f_j(\theta,x) = \frac{1}{|x|^3} \int_0^{2\pi} d\varphi \int_0^\infty F_x[u_j] \exp\left[-i\tau\left(\frac{x_3}{|x|}\cos\theta + \frac{r}{|x|}\sin\theta\cos\varphi\right)\right] \tau^2\, d\tau \tag{4.6}$$

Applying the method of stationary phase we obtain in this case from (4.5) the following asymptotic formula:

$$u(x,t) \sim (2\pi)^{-5/2} (Nt)^{-1/2} [f_0(\pi/2, x) \cos(Nt - \pi/4) + N^{-1} f_1(\pi/2, x) \sin(Nt - \pi/4)] \tag{4.7}$$

Large Nt and fixed $\omega = |x|/(dNt)$. Using again formula (4.1) and introducing new variables for integration β, θ, φ by formulas which differs from (4.4) in that βNt has been substituted for τ, we write (4.1) as follows:

$$\begin{gathered}u(x,t) = \int_{-\pi/2}^{3\pi/2} d\varphi \int_0^\infty \beta^2 d\beta \int_0^\pi [Q_1' \exp(iNt\Psi_1) + Q_2' \exp(iNt\Psi_2)]\, d\theta \\ Q_j' = 2^{-4} (\pi d\omega)^{-3} \{F_x[u_0] \sin\theta + iN^{-1} (-1)^j F_x[u_1]\} \\ \Psi_j = (-1)^{j+1} \sin\theta - \beta(x_3/|x| \cos\theta + r|x|^{-1} \sin\theta\cos\varphi)\end{gathered} \tag{4.8}$$

We apply the method of stationary phase /11/ for multiple integrals in the analysis of the triple integrals in (4.8).

Having solved the equations

$$\operatorname{grad}\Psi_j = 0, \quad j = 1, 2$$

we find the points of stationary state with the following values of variables φ, β, θ:

$$\varphi_0 = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \quad \theta_0 = \frac{\pi}{2}, \quad \beta_0 = 0 \text{ for } \Psi_1 \text{ and } \Psi_2 \tag{4.9}$$

$$\varphi_1 = 0, \ \theta_1 = \frac{\pi}{2}(1 + \operatorname{sgn} x_3) - \arcsin\frac{x_3}{|x|}, \quad \beta_1 = \frac{r}{|x|} \text{ for } \Psi_1 \tag{4.10}$$

$$\varphi_2 = \pi, \ \theta_2 = \frac{\pi}{2}(1 - \operatorname{sgn} x_3) + \arcsin\frac{x_3}{|x|}, \quad \beta_2 = \frac{r}{|x|} \text{ for } \Psi_2 \tag{4.11}$$

Assuming that functions $F_x[u_0]$ and $F_x[u_1]$ do not have singularities when $\beta = 0$, we can neglect the contribution of points (4.9) to the value of integral (4.8) because of the presence of cofficient β^2 in the integrands. Consequently, we calculate only the contribution of points (4.10) and (4.11) for which we have

$$\begin{gathered}\Psi_j = (-1)^{j+1} |x_3| / |x| \\ |\det \operatorname{Hess}\Psi_j| = |x_3| r^2 |x|^{-3}, \quad \operatorname{sgn} \|\operatorname{Hess}\Psi_j\| = (-1)^{j+1}\end{gathered}$$

where $\operatorname{Hess}\Psi_j$ is the Hessian matrix of functions Ψ_j.

Taking into account the last formulas we find that as $Nt \to \infty$ with fixed ω the principal term of the asymptotic expansion of integral (4.8) is of the form

$$\begin{gathered}u(x,t) \sim (2\pi Nt)^{-1/2} (\omega d)^{-3} r (|x_3| \cdot |x|)^{-1/2} \\ \{|x_3| \cdot |x|^{-1} (\operatorname{Re} F_x'[u_0] S_1 - \operatorname{Im} F_x'[u_0] S_2) + \\ N^{-1} (\operatorname{Re} F_x'[u_1] S_2 + \operatorname{Im} F_x'[u_1] S_1)\} \\ S_1 = \cos(Nt|x_3|/|x| + \pi/4), \quad S_2 = \sin(Nt|x_3|/|x| + \pi/4)\end{gathered} \tag{4.12}$$

where $\operatorname{Re} F_x'[u_j]$ and $\operatorname{Im} F_x'[u_j]$ ($j = 0, 1$) are, respectively, the real and imaginary parts of functions $F_x[u_j]$ at point (4.10) on the supplementary assumption that $x_3 \neq 0$.

The derived formulas (4.3), (4.7), and (4.12) give a hydrodynamic picture capable of fairly clear interpretation which enables us to come to a number of conclusions relative to the process propagation of initial perturbations of a perfect incompressible continuously stratified fluid with constant Brent– Väisälä frequency.

Formula (4.3) implies that in the distant zone the internal wave field generated by initial perturbations occurring in some region B, have the same structure as the field defined by the fundamental solution $E(x, t)$ of the internal wave operator. The difference concerns the wave amplitude which is indicated by the presence in (4.3) of amplitude multipliers equal to integrals of functions $\Delta_3 u_0(x)$ and $\Delta_3 u_1(x)$ taken over region B.

This conclusion is valid, as seen from estimate (4.2), only from fairly large relations $|x|/d$, which increases with increasing time from the instant of initial perturbation action to that of observation, and the relation $|x|/d$ should also increase.

Formulas (4.7) and (4.12) enable us to assess the nature of the process for large values of the dimensionless time Nt.

It follows from (4.12) that in the case of large dimensional time for observation points at distances $|x| = \omega dNt$, where ω is fixed and nonzero, from the coordinate origin, progressive waves similar to those described in /1/ propagate in the fluid in fairly distant regions of space. Singularity of these waves is in that they appear to radiate at frequency N from vertical semiaxes $x_3 > 0$ and $x_3 < 0$ are then absorbed by the horizontal plane $x_3 = 0$; the equal phase surfaces of these waves coincide with the conical surfaces $|x_3| / |x| = \text{const}$. The angular velocity of these surfaces is $x_3 t^{-1} (x_1^2 + x_2^2)^{-1/2}$.

Similar waves were observed in laboratory experiments /12/ where internal waves were induced in linearly stratified fluid by the initial perturbation concentrated in very small region. It is interesting to note that the photographs b and c of Fig.2 in /12/ show that regular wave systems recede more and more from the initial perturbation region with increasing time.

It follows from (4.7) that in the later stages of development of the process in a fixed observation area, fluid motions are of the standing internal wave type with the Brent–Väisälä frequency. The amplitude variation of these waves in space is defined by formula (4.6) from which it is difficult to draw any conclusions without specific definition of initial functions $u_0(x)$ and $u_1(x)$.

With increasing time the amplitude of standing waves decreases as $(Nt)^{-1/2}$, while at the same time the solution of the Cauchy problem for Sobolev's equation diminishes with passing time as t^{-1} /5/.

The conclusions based formula (4.7) must, obviously, be treated with caution, since during the later stages of a real process the effect of viscosity, which is not taken into account in this work, considerably increases.

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